Madras College Maths Department
Higher Maths
E&F 1.2 Trigonometric Expressions

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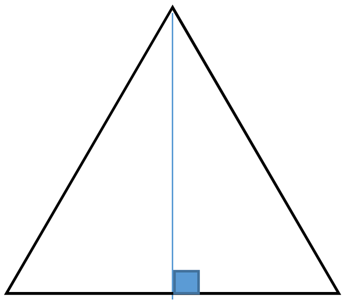
Written solutions for each exercise are available at http://madrasmaths.com/courses/higher/revison_materials_higher.html

You should check your solutions at the end of each exercise and ask your teacher or attend study support if there any problems.

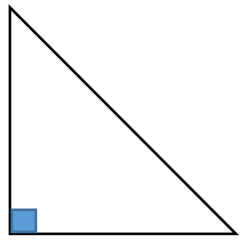
Exact Values

Exact values of sin, cos or tan of an angle involve surds and fractions rather than decimal values from a calculator which often need to be rounded off. Exact values of 0^o, 30^o, 45^o, 60^o, 90^o and angles related to these should be known.
For 0^o, 90^o, 180^o etc, use the graphs of the functions to remember the values.

For angles of 30^o and 60^o use an equilateral triangle of side 2.



For angles of 45^o use a right-angled Isosceles triangle of side 1.

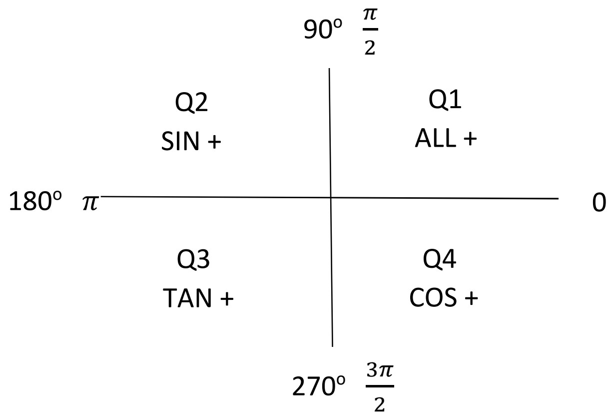


EXACT VALUES

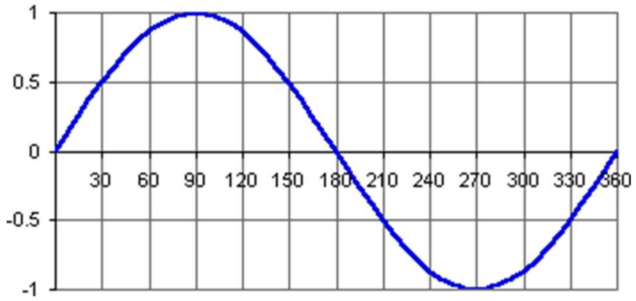
L	A in deg	0^o	30^o	45^o	60^o	90^o	L
E	in rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	E
A	Sin A						A
R	Cos A						R
N	Tan A						N

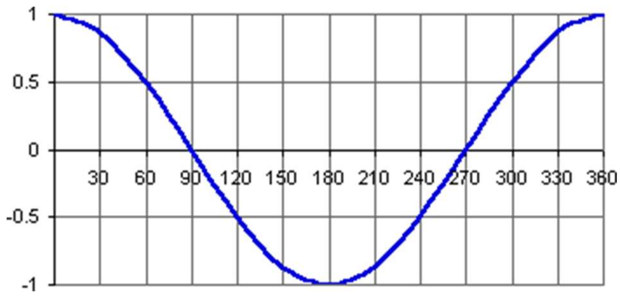
Related Angles

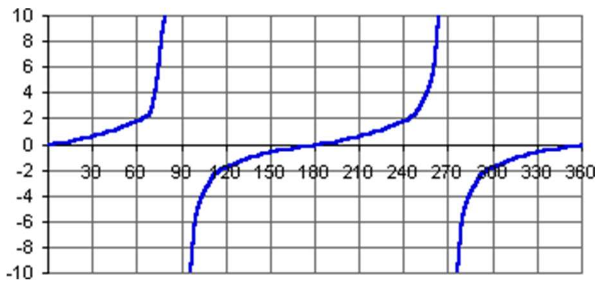
Using quadrants:
Angles are normally measured clockwise from the positive x axis.
Negative angles are measured anticlockwise.
The related acute angle is the angle from the x axis (horizontal)



Alternatively, the related angles can be found from the graphs.







Examples:
Find the related angles and exact values of:

(1) $\sin 240^{o}$

(2) $\cos(-45^{o})$

(3) $\tan 495^{o}$

(4) Given that $\cos A = \frac{2}{3}$, find the exact values of sin A and tan A.

Your questions:

(1) $\cos 120^{o}$

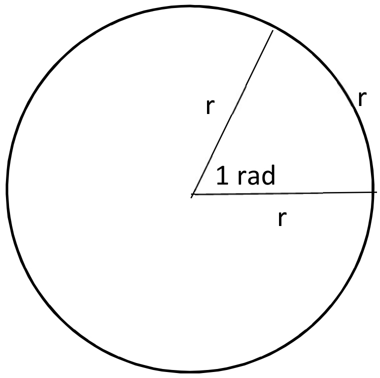
(2) $\tan(-60^{o})$

(3) $\sin 675^{o}$

(4) Given that $\sin A = \frac{2}{5}$, find the exact values of cos A and tan A.

Radian Measure

Angles can be measured in degrees or in radians.
Radian measure of an angle is defined as



$$\theta = \frac{arc\ length}{radius}$$

So 1 radian is the angle subtended at the centre of a circle from an arc equal in length to the radius.
In a full turn of 360° there are 2π radians since C = 2 πr.

Learn: 2π radians = 360°

Or more simply π radians = 180°

It is now extremely important that the degrees symbol is used if working in degrees.

Converting Degrees → Radians

- multiply by $\frac{\pi}{180}$

(1) 45°

(2) 150°

(3) 270°

- use simple proportion

Deg	Rad
180°	π
45 °	
30 °	
150 °	
90 °	
270 °	

Additional examples are available in the Dynamic Maths Study Notes available at http://madrasmaths.com/courses/higher/revison_materials_higher.html (password: madrasmaths) and at hsn.uk.net

Converting Radians ➔ Degrees

- multiply by $\frac{180}{\pi}$

(1) $\frac{\pi}{3}$ rad

(2) $\frac{4\pi}{5}$ rad

(3) $\frac{11\pi}{6}$ rad

- use simple proportion

Rad	Deg
π	180°
$\frac{\pi}{3}$	
$\frac{4\pi}{5}$	
$\frac{11\pi}{6}$	

1) Find the exact values of:

a) $\sin\frac{\pi}{3} =$

b) $cos\frac{3\pi}{4} =$

Your questions

a) $\cos\frac{\pi}{4} =$

b) $tan\frac{5\pi}{4} =$

Additional examples are available in the Dynamic Maths Study Notes available at http://madrasmaths.com/courses/higher/revison_materials_higher.html (password: madrasmaths) and at hsn.uk.net

Addition Formulae

The Addition formulae are required to deal with sin or cos of angles which are expressed as the sum or difference of two angles. Although these formulae are given in the exam, they are worth remembering if possible. They are needed to express a compound angle in expanded form and also to reduce the expanded form back to a single trig function.

There are four formulae used in higher:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$
$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

and are given on the formula sheet as follows:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

read the signs consistently along the top or bottom of the formula

Examples:

1. Rewrite these expressions in terms of a single function

(a) Sin x cos 45° + cos x sin 45°

(b) $cos\frac{\pi}{2}\ cos\frac{\pi}{6} + \ sin\frac{\pi}{2}\ sin\frac{\pi}{6}$

2. Expand these compound angles and simplify using exact values.

(a) $\text{Sin}\,(x - 30)^{\circ}$ =

(b) $\text{Cos}\,(A + \frac{2\pi}{3})$ =

3. Given that P and Q are both acute angles and $\sin P = \frac{4}{5}$ and $\cos Q = \frac{8}{17}$, find the exact value of $\sin(P + Q)$.

(Here we need to have the values of sin P and cos P , sin Q and cos Q to substitute into the addition formula for $\sin(P + Q)$. The unknown values can be found either by the formula $\cos^2 A + \sin^2 A = 1$ or by completing right angled triangles using Pythagoras Theorem).

4. A is the point (3, 4) and B is the point (15, 8). Find the exact value of cosAOB.

Double Angle Formulae

If the same angle is considered for both A and B in the addition formulae, then the following formulae can be derived for double angles.

$$\begin{aligned}\sin 2A &= 2\sin A\cos A\\ \cos 2A &= \cos^2 A - \sin^2 A\\ &= 2\cos^2 A - 1\\ &= 1 - 2\sin^2 A\end{aligned}$$

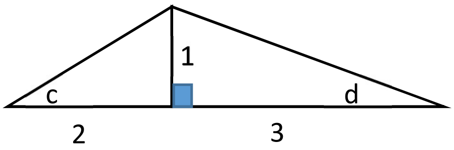
Examples:

1. Express cos 10x in terms of
   (a) cos 5x and sin 5x

   (b) sin 5x only

2. Using the double angle formulae, find the exact value of $2\sin 165^{\circ}\cos 165^{\circ}$

3. EXAM QUESTION
The diagram shows two right-angled triangles with angles c and d marked as shown.



(a) Find the exact value of sin (c + d).
(b) (i) Find the exact value of sin 2c.
(ii) Show that cos 2d has the same exact value.

Trigonometric Identities

When proving a trig identity we must

- separate the LHS from the RHS
- work through the LHS independently until we get the result on the RHS
- or should that prove difficult, work through both sides independently until they are equal.

Recap on formulae which may be needed to prove an identity.

For any angle A		
$\sin(-A) = -\sin A$ $\cos(-A) = \cos A$ $\tan(-A) = -\tan A$	$\sin(90 - A) = \cos A$ $\cos(90 - A) = \sin A$	$\sin^2 A + \cos^2 A = 1$ $\tan A = \dfrac{\sin A}{\cos A}$
For any angles A and B		
$\sin(A + B) = \sin A \cos B + \cos A \sin B$ $\sin(A - B) = \sin A \cos B - \cos A \sin B$ $\cos(A + B) = \cos A \cos B - \sin A \sin B$ $\cos(A - B) = \cos A \cos B + \sin A \sin B$	$\sin 2A = 2\sin A\cos A$	$\cos 2A = \cos^2 A - \sin^2 A$ $= 2\cos^2 A - 1$ $= 1 - 2\sin^2 A$

Examples:

1. Prove that $\cos(270 + a)^\circ = \sin a^\circ$

2. Prove that $\frac{\cos(a-b)}{\cos a \cos b} = 1 + \tan a \tan b$

3. Prove that $(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 2[1 + \cos(x + y)]$

Wave function

To express a function in the form acos x + bsin x in the form kcos $(x - \alpha)$

- Expand kcos $(x - \alpha)$ using the addition formula
  acos x + bsin x = kcos $(x - \alpha)$
  = kcos x cos α + ksin x sin α

- Equate coefficients of cos x and sin x - these equations must be stated clearly
  a = kcos α, b = ksin α

- Find k
  $a^2 = k^2\cos^2\alpha \qquad b^2 = k^2\sin^2\alpha$
  $a^2 + b^2 = k^2\cos^2\alpha + k^2\sin^2\alpha$
  $a^2 + b^2 = k^2(\cos^2\alpha + \sin^2\alpha)$
  $a^2 + b^2 = k^2$ since $\cos^2\alpha + \sin^2\alpha = 1$
  so k = $\boldsymbol{\sqrt{a^2 + b^2}}$ **LEARN RESULT**

Find α
$\boldsymbol{\tan\alpha = \frac{k\sin\alpha}{k\cos\alpha}}$
the quadrant for α is determined by the signs of ksin α and kcos α
LEARN METHOD

Examples:

1. Express $\sqrt{3}cosx° - sinx°$ in the form kcos $(x - \alpha)$ where k > 0 and $0 \leq \alpha \leq 360°$

$$\sqrt{3}cosx° - sinx° =$$

Other forms of the wave function

Expressing $a\cos x + b\sin x$ in the form $k\cos(x \pm \alpha)$ or $k\sin(x \pm \alpha)$ is done in the same way as for $k\cos(x - \alpha)$ but more care may be needed in equating the coefficients.

Examples:

2. Express $4\cos x^\circ + 3\sin x^\circ$ in the form $r\sin(x - \alpha)^\circ$ where $r > 0$ and $0 \le \alpha \le 360°$

Additional examples are available in the Dynamic Maths Study Notes available at http://madrasmaths.com/courses/higher/revison_materials_higher.html (password: madrasmaths) and at hsn.uk.net

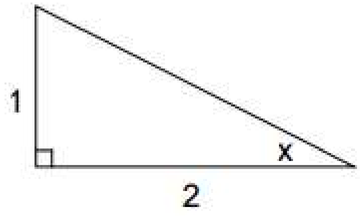
Your question

3. Express $\sqrt{3}\, cos\theta + \ sin\theta$ in the form r $\sin(\theta + \ \alpha)$ where r > 0 and 0 $\leq \ \alpha \ \leq 2\pi$

Practice unit assessment questions

Practice A

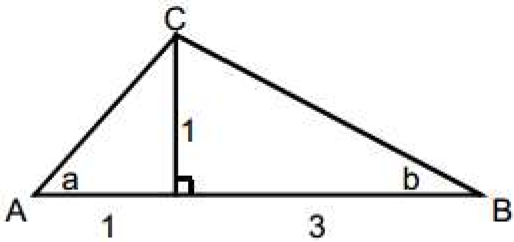
1. Given that x is an acute angle and tan x = $\frac{1}{2}$,
cos 2x will be equal to



A $\frac{3}{5}$ B $-\frac{3}{5}$ C $\frac{3}{\sqrt{5}}$ D $-\frac{3}{\sqrt{5}}$

(3)

2. In triangle ABC, show that the exact value of
sin(a + b) is $\frac{2}{\sqrt{5}}$.



(3)

3. Express $4\cos x^\circ + \sin x^\circ$ in the form $k\cos(x - a)^\circ$ where $k > 0$ and $0 \le a < 360$

Calculate the values of *k* and *a.*

(4)

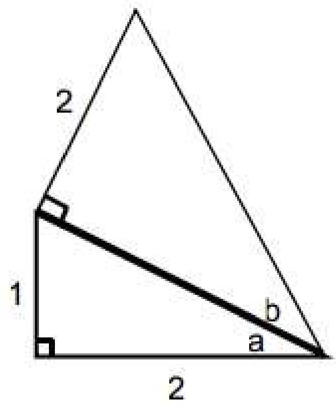
Additional examples are available in the Dynamic Maths Study Notes available at http://madrasmaths.com/courses/higher/revison_materials_higher.html (password: madrasmaths) and at hsn.uk.net

Practice B

1.

For the diagram opposite, show that

$\cos(a + b)$ is $\dfrac{2\sqrt{5} - 2}{3\sqrt{5}}$.



(4)

2. Given $\tan x = \dfrac{1}{7}$, show that $\sin 2x$ is $\dfrac{7}{25}$.

(3)

3. Express the following in the form *k sin(x + a)*° where
k > 0 and 0 ≤ *a* < 2π for 6 sin x + 8 cos x. State the values of *k and a.*

(4)

Additional examples are available in the Dynamic Maths Study Notes available at http://madrasmaths.com/courses/higher/revison_materials_higher.html (password: madrasmaths) and at hsn.uk.net

Higher Maths Homework – Trigonometry

Attempt all questions. Do not leave any blanks!

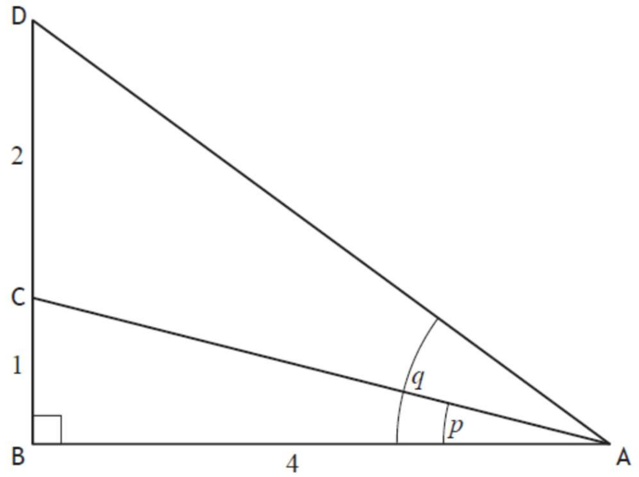
Video help is available at YouTube.com/DLBMaths or search for e.g. YouTube DLBMaths SQA Higher Maths 2012 Question 7

Once you have **completed and marked** your homework using the videos above please grade yourself on each question using the following code:

1 – fully understood and completed on own
2 – partially understood and now understand after using video help
3 – looked at video help, copied down the solution but will need extra help from my teacher.

Non-Calculator Section

1 Triangle ABD is right-angled at B with angles BAC $= p$ and BAD $= q$ and lengths as shown in the diagram below.



Show that the exact value of $\cos(q-p)$ is $\frac{19\sqrt{17}}{85}$. 5

SQA Higher Maths 2016 Paper 1 Question 13

2

Given that $\tan 2x = \frac{3}{4}$, $0 < x < \frac{\pi}{4}$, find the exact value of

(a) $\cos 2x$ **1**

(b) $\cos x$. **2**

SQA New Higher Maths 2015 Paper 1 Question 10

3 (*a*) Using the fact that $\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$, find the exact value of $\sin\left(\frac{7\pi}{12}\right)$. **3**

(*b*) Show that $\sin(A + B) + \sin(A - B) = 2\sin A\cos B$. **2**

(*c*) (i) Express $\frac{\pi}{12}$ in terms of $\frac{\pi}{3}$ and $\frac{\pi}{4}$.

(ii) Hence or otherwise find the exact value of $\sin\left(\frac{7\pi}{12}\right) + \sin\left(\frac{\pi}{12}\right)$. **4**

SQA Higher Maths 2009 Paper 1 Question 24

<u>Calculator Section</u>

4 Express $5\cos x - 2\sin x$ in the form $k\cos(x + a)$,
where $k > 0$ and $0 < a < 2\pi$. **4**

SQA Higher Maths 2016 Paper 2 Question 8(a)

5 Scientists are studying the growth of a strain of bacteria. The number of bacteria present is given by the formula

$$B(t) = 200e^{0\cdot107t},$$

where t represents the number of hours since the study began.

(a) State the number of bacteria present at the start of the study. **1**

(b) Calculate the time taken for the number of bacteria to double. **4**

SQA Higher Maths 2016 Paper 2 Question 6

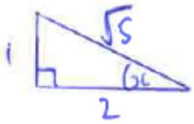
<u>Check list:</u>

Attempted all questions.
Checked each question and graded 1, 2 or 3 as described above.
Copied out any solution that I couldn't get so that I can discuss with a teacher.

Practice Unit Assessment Solutions

Practice A

1

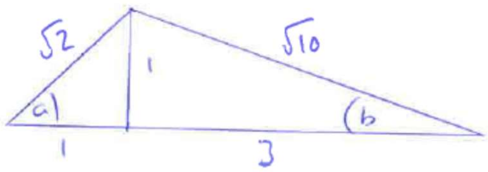


$\sin x = \frac{1}{\sqrt{5}}$

$\cos x = \frac{2}{\sqrt{5}}$

$$\begin{aligned} &\cos 2x \\ &= \cos^2 x - \sin^2 x \\ &= \left(\frac{2}{\sqrt{5}}\right)^2 - \left(\frac{1}{\sqrt{5}}\right)^2 \\ &= \frac{4}{5} - \frac{1}{5} \\ &= \frac{3}{5} \end{aligned}$$

2



$\sin a = \frac{1}{\sqrt{2}}$ $\sin b = \frac{1}{\sqrt{10}}$

$\cos a = \frac{1}{\sqrt{2}}$ $\cos b = \frac{3}{\sqrt{10}}$

$$\begin{aligned} \sin(a+b) &= \sin a \cos b + \cos a \sin b \\ &= \frac{1}{\sqrt{2}} \cdot \frac{3}{\sqrt{10}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{10}} \\ &= \frac{4}{\sqrt{20}} \\ &= \frac{4}{\sqrt{4}\sqrt{5}} \\ &= \frac{4}{2\sqrt{5}} \\ &= \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} \\ &= \frac{2\sqrt{5}}{5} \end{aligned}$$

(3) $4\cos x + \sin x = k\cos(x-a)$

$= k(\cos x\cos a + \sin x\sin a)$

$4\cos x + \sin x = k\cos x\cos a + k\sin x\sin a$ ✓

$k\cos a = 4$

$k\sin a = 1$ ✓

$k^2 = 4^2 + 1^2$

$k^2 = 17$

$k = \sqrt{17}$

$\tan a = \dfrac{k\sin a}{k\cos a}$

$\tan a = \dfrac{1}{4}$

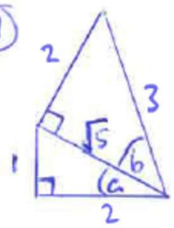
$a = 14.0°$

$k\sin a$ is +ve

$k\cos a$ is +ve

Practice Test B

B (1)



$\cos a = \dfrac{2}{\sqrt{5}}$

$\sin a = \dfrac{1}{\sqrt{5}}$

$\cos b = \dfrac{\sqrt{5}}{3}$

$\sin b = \dfrac{2}{3}$

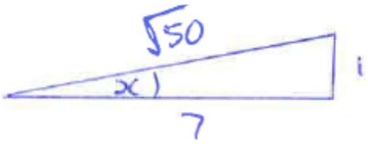
$\cos(a+b) = \cos a\cos b - \sin a\sin b$

$= \dfrac{2}{\sqrt{5}} \cdot \dfrac{\sqrt{5}}{3} - \dfrac{1}{\sqrt{5}} \cdot \dfrac{2}{3}$

$= \dfrac{2\sqrt{5}}{3\sqrt{5}} - \dfrac{2}{3\sqrt{5}}$

$= \dfrac{2\sqrt{5} - 2}{3\sqrt{5}}$

② $\tan x = \frac{1}{7}$



$\sin x = \frac{1}{\sqrt{50}}$

$\cos x = \frac{7}{\sqrt{50}}$

$$\begin{aligned}\sin 2x &= 2\sin x\cos x \\ &= 2 \cdot \frac{1}{\sqrt{50}} \cdot \frac{7}{\sqrt{50}} \\ &= \frac{14}{50} \\ &= \underline{\underline{\frac{7}{25}}}\end{aligned}$$

③ ~~$K\sin(x+a) = K\sin$~~

$$\begin{aligned}6\sin x + 8\cos x &= K\sin(x+a) \\ &= K(\sin x\cos a + \cos x\sin a) \\ 6\underline{\sin x} + 8\underline{\underline{\cos x}} &= K\underline{\sin x}\cos a + K\underline{\underline{\cos x}}\sin a \quad \cdot\checkmark\end{aligned}$$

$K\cos a = 6$

$K\sin a = 8 \quad \cdot\checkmark$

$K^2 = 6^2 + 8^2$

$K^2 = 100$

$\underline{\underline{K = 10}}$

$\tan a = \frac{K\sin a}{K\cos a}$

$\tan a = \frac{6}{8}$

$a = 0.643\ldots$

$\underline{\underline{a = 0.64 \text{ radians}}}$

$\sin a$ is +ve

S ✓	A ✓✓
T	C ✓

$\cos a$ is +ve

make sure you see if it is a radian or degree question.